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P. Turán: Reminiscences of His Student

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P. Turán died on September 26, 1976, at the age of 66. As his former student and collaborator I would like to write about him and his work.

He was born on August 28, 1910, and received his Ph.D. under L. Fejér in 1935. He was without a regular position until 1938, when he became a high-school teacher. In 1949 he became a professor at the University of Budapest. Earlier he spent some time in Copenhagen, and a year at the Institute for Advanced Study in Princeton. The international recognition of his mathematical achievements resulted in many invitations, and he was a frequent lecturer and visiting professor at western universities. The last time I saw him was in the summer of 1975 in Montreal, where he gave a series of lectures on unsolved problems in approximation theory at the University of Montreal. Although he was unaware of the fact, he was already terminally ill. His voice sounded tired but his interest in mathematics was as lively as ever.

His enthusiasm for mathematics, his versatility, and his knowledge were overwhelming. One could hardly imagine a situation where he would not think about mathematics. A long paper on Riemann's ζ-function (see [31] of his List of Publications) was conceived (and, at least partly, written) during his confinement in a Nazi forced labor camp. Turán used to hold, during the summer, seminars at a swimming-pool, and we two had long mathematical conversations during strenuous rowing excursions on the Danube. On one occasion, in 1960, he swam across the Danube, a non-trivial achievement; afterwards he remarked that he celebrated his fiftieth birthday in addition to writing a good paper, and rejoicing over the birth of his third son.

The mathematical work of Paul Turán left a lasting impact on graph theory, analytic number theory, trigonometric series and approximation theory. I shall confine myself to a few remarks on some of his achievements.

Turán's Ph.D. thesis was on a theorem of Hardy and Ramanujan according to which, in some sense, "most" of the natural numbers $\leqq N$ have "about" $\log\log N$ prime divisors, with or without counting multiplicity. Their proof was based on a generalization of the prime number theorem, and was very complicated.

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Turán was not familiar at the time with the classical Cebyshev inequality of probability theory; he rediscovered it and used it to give an amazingly simple proof of the above theorem. This was the first time that probabilistic methods were used to solve a problem of number theory.

In 1959, together with Renyi [95], he returned to this question. Combining methods of probability theory and analytic number theory, they gave a simplified proof of the refinement of Hardy and Ramanyjan's theorem due to Erdös and Kac, and solved a hitherto open problem in this area.

The joint work of Turán and S. Knapowski, which was carried on for several years, should also be mentioned. Their work deals with the delicate question of what they termed "Comparative Prime-Number Theory." The problem was to find inequalities for the difference $\pi(x, k, l_1) - \pi(x, k, l_2)$, where $\pi(x, k, l)$ is the number of primes $\leqslant x$ which are $\equiv l (\mathrm{mod}\ k)$.

In my judgement Turán's most significant work is his power-sum method. The method consists of "localizing" the positive integral solutions t of inequalities of the types

$$F(t) = |a_1 z_1^t + a_2 z_2^t + a_n z_n^t| > K \max |z_k|^t.$$

His results are described in his book [67]. A new English edition is in preparation.

The power-sum method is extremly powerful. Its applications range over estimates on Riemann's ζ function, a new proof of Fabry's gap theorem, differential equations, and even the numerical solution of algebraic equations, where a deficiency in Graeffe's method can be corrected.

Since a description of some of Turán's work on interpolation and approximation appears elsewhere in this issue I shall not touch on this matter.

Turán was always ready to help people who needed him. His untimely death was not a surprise; he had been ill for several years. I am sure that I speak for all of his students when I say that his death was a personal loss for us. Paul Turán lives on in his work and in the work of his students.